On Non-Scattering Energies

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Doctoral dissertation

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Contents

Abstract

In this thesis, we extend the theory of non-scattering energies on two fronts. First, we shall consider the discreteness of non-scattering energies corresponding to non-compactly supported potentials using the approach via transmission eigenvalues and fourth-order operators. The method requires the support of the potential to exhibit certain compact Sobolev embedding and to be contained in a half-space and the potential to have controlled polynomial or exponential decay at infinity. Also, in order to connect the non-scattering energies to the fourth-order operators, a generalization of the classical Rellich theorem to unbounded domains is required. This is of independent interest, and we obtain several such results, including a discrete analogue.

Our second contribution (joint work with L. Päivärinta and M. Salo) is extending a recent result on non-existence of non-scattering energies for potentials with rectangular corners to arbitrary corners of angle smaller than π in two dimensions, and to prove in three dimensions that the set of strictly convex circular conical corners for which non-scattering energies might exist is at most countable.

This thesis consists of the papers

I. Vesalainen, E. V.: *Transmission eigenvalues for non-compactly supported potentials*, Inverse Problems, 29 (2013), 104006, 1–11.

II. Vesalainen, E. V.: *Rellich type theorems for unbounded domains*, to appear in Inverse Problems and Imaging. Preprint available at arXiv:1401.4531 [math.AP].

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The author of this thesis had an equal role in the research and writing of the joint article.

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Introduction

1 Scattering and non-scattering energies

1.1 Scattering theory

Our objects of study arise from scattering theory. More precisely, time independent scattering theory for short-range potentials, which models e.g. two-body quantum scattering, acoustic scattering, and some classical electromagnetic scattering situations (for a general reference, see e.g. [10]). Here one is concerned with the situation where, at a fixed energy or wavenumber $\lambda \in \mathbb{R}_+$, an incoming wave w, which is a solution to the free equation

$$(-\Delta - \lambda)\, w = 0,$$

is scattered by some perturbation of the flat homogeneous background. Here this perturbation will be modeled by a real-valued function V in $\mathbb{R}^n$ having enough decay at infinity. The total wave v, which models the "actual" wave, then solves the perturbed equation

$$(-\Delta + V - \lambda)\, v = 0.$$

In acoustic and electromagnetic scattering, one has λV instead of V. Of course, the two waves v and w must be linked together and the connection is given by the Sommerfeld radiation condition. The upshot will be that the difference u of v and w, the so-called scattered wave, will have an asymptotic expansion of the shape

$$u(x) = v(x) - w(x) = A\left(\frac{x}{|x|}\right) \frac{e^{i\sqrt{\lambda}|x|}}{|x|^{(n-1)/2}} + \text{error},$$

where A depends on λ and w, and where the error term decays more rapidly than the main term. The point here is that in the main term the dependences on the radial and angular variables are neatly separated, and in practical applications one usually measures the scattering amplitude or far-field pattern A, or its absolute value $|A|$.

1.2 Non-scattering energies

It is a natural question whether we can have $A \equiv 0$ for some $w \not\equiv 0$? This would mean that the main term of the scattered wave vanishes at infinity, meaning that the perturbation, for the special incident wave in question, is not seen far away. Values of $\lambda \in \mathbb{R}_+$ for which such an incident wave w exist, are called non-scattering energies (or appropriately, wavenumbers) of V.

Of course, the functions u, v and w will be from some specific function spaces. To be precise, $\lambda \in \mathbb{R}_+$ is a non-scattering energy for a short-range potential V if there exist non-zero functions $v, w \in B_2^*$ solving the equations

$$\begin{cases} (-\Delta + V - \lambda)\, v = 0, \\ (-\Delta - \lambda)\, w = 0, \end{cases}$$

in $\mathbb{R}^n$, and having the same asymptotic behaviour at infinity in the sense that $u = v - w \in \mathring{B}_2^*$. Here

$$B_2^* = \big\{ u \in B^* \;\big|\; \partial^\alpha u \in B^* \text{ for multi-indices } \alpha \text{ with } |\alpha| \leqslant 2 \big\},$$

and similarly

$$\mathring{B}_2^* = \big\{ u \in \mathring{B}_2^* \;\big|\; \partial^\alpha u \in \mathring{B}_2^* \text{ for multi-indices } \alpha \text{ with } |\alpha| \leqslant 2 \big\},$$

where B^* and $\mathring{B}^*$ are the Agmon–Hörmander spaces

$$B^* = \left\{ u \in L^2_{\mathrm{loc}}(\mathbb{R}^n) \;\middle|\; \sup_{R>1} \frac{1}{R} \int\limits_{B(0,R)} |u|^2 < \infty \right\}$$

and

$$\mathring{B}^* = \left\{ u \in B^* \;\middle|\; \lim_{R \longrightarrow \infty} \frac{1}{R} \int\limits_{B(0,R)} |u|^2 = 0 \right\}.$$

A function $V \in L^2_{\mathrm{loc}}(\mathbb{R}^n)$ is a short-range potential for instance when $V(\cdot) \ll \langle \cdot \rangle^{-\alpha}$ in $\mathbb{R}^n$ for some $\alpha \in \,]1, \infty[$. For a presentation of short-range scattering theory in terms of the Agmon–Hörmander spaces, see e.g. Chapter XIV of [18] and the first sections of [30]. Also, the articles [3] and [4] are recommended.

Results on the existence of non-scattering energies are scarce. Essentially only two general results are known: For compactly supported radial potentials the set of non-scattering energies is an infinite discrete set accumulating at infinity [11], and for compactly supported potentials with rectangular corners, Blåsten, Päivärinta and Sylvester recently proved that the set of non-scattering energies is empty [5]. In addition to these, the many results on the discreteness of transmission eigenvalues for various compactly supported potentials also imply the discreteness of non-scattering energies for some of the corresponding potentials. It is not yet known if non-scattering energies can exist for non-radial potentials.

We would like to mention the related topic of transparent potentials: there one considers (at a fixed energy) potentials for which A vanishes for all w. The knowledge of transparent potentials is more extensive. In particular, several constructions of such radial potentials have been given, see e.g. the works of Regge [31], Newton [28], Sabatier [35], Grinevich and Manakov [13], and Grinevich and Novikov [14].

2 Discreteness via fourth-order operators

2.1 The compactly supported story

Discreteness of the set of non-scattering energies tends to be a much more attainable goal than knowledge of existence or non-existence. The first key step towards that goal (for compactly supported V) is supplied by Rellich's classical uniqueness theorem which is the following:

Theorem 1. *Let $u \in \mathring{B}_2^*$ solve the equation $(-\Delta - \lambda)\, u = f$, where $\lambda \in \mathbb{R}_+$ and $f \in L^2(\mathbb{R}^n)$ is compactly supported. Then u also is compactly supported.*

This was first proved (though with a bit different decay condition) independently by Rellich [32] and Vekua [43] in 1943. Of the succeeding work, which includes generalizations of this result to more general constant coefficient differential operators, we would like to mention the work of Trèves [41], Littman [24, 25, 26], Murata [27] and Hörmander [17]. Section 8 of [16] is also interesting.

Now, assume that V is compactly supported. The equations for v and w imply that the scattered wave u solves the equation

$$(-\Delta - \lambda)\, u = -Vv.$$

If furthermore $A \equiv 0$, then u will satisfy the decay condition in Theorem 1, and so $u = v - w$ will vanish outside a compact set. If the support of V is essentially contained in some suitable open domain Ω, the unique continuation principle for the free Helmholtz equation allows us to conclude that actually

$$\begin{cases} (-\Delta + V - \lambda)\, v = 0 & \text{in } \Omega, \\ (-\Delta - \lambda)\, w = 0 & \text{in } \Omega, \\ v - w \in H_0^2(\Omega). \end{cases}$$

This system, called the interior transmission problem, is a non-self-adjoint eigenvalue problem for λ, and the values of λ, for which this system has non-trivial L^2-solutions, are called (interior) transmission eigenvalues.

The non-scattering energies and transmission eigenvalues first appeared in the papers of Colton and Monk [11] and Kirsch [22]. In [9] Colton, Kirsch and Päivärinta proved the discreteness of transmission eigenvalues (and non-scattering energies) for potentials that may even be mildly degenerate. The early papers on the topic also considered, among other things, radial potentials; for more on this, we refer to the article of Colton, Päivärinta and Sylvester [12].

In recent years, there has been a surge of interest in the topic starting with the general existence results of Päivärinta and Sylvester [30], who established existence of transmission eigenvalues for a large class of potentials, and Cakoni, Gintides and Haddar [6], who established for acoustic scattering, that actually the set of transmission eigenvalues must be infinite.

For potentials more general than the radial ones, a very common approach to proving discreteness and other properties has been via quadratic forms: the scattered wave solves the fourth-order equation

$$(-\Delta + V - \lambda)\, \frac{1}{V}\, (-\Delta - \lambda)\, u = 0,$$

and this can be handled nicely with quadratic forms (or with variational formulations) and analytic perturbation theory. We shall discuss this in more detail below.

Recently, other more general approaches, not involving the fourth-order equation, to proving discreteness and many other results have been introduced by Sylvester [39], Robbiano [33], and Lakshtanov and Vainberg [23].

For more information on transmission eigenvalues, we recommend the survey of Cakoni and Haddar [7] and their editorial [8] as well as the articles mentioned there and their references.

2.2 Non-compactly supported potentials

Most of the work on non-scattering energies and transmission eigenvalues deals with compactly supported potentials V. However, the basic short-range scattering theory only requires V to have enough decay at infinity, essentially something like $V(x) \ll \langle x \rangle^{-1-\varepsilon}$. Thus, it makes perfect sense to study non-scattering energies for non-compactly supported potentials.

In [44, 45] we take first steps into the direction of non-compact supports by considering non-scattering energies and transmission eigenvalues for non-compact Ω which are nearly compact in the sense that they have a suitable compact Sobolev embedding, and for potentials V taking only positive real values and having a certain kind of controlled asymptotic behaviour. For these potentials, we prove the basic discreteness result using the approach via fourth-order operators described above. The more usual case of bounded Ω with a positive real-valued potential, which is bounded and bounded away from zero, is covered as a special case. The potential V may decay polynomially or exponentially fast at infinity. The latter case is simpler, and in the following we shall focus on it. The precise statement of the result is the following.

Theorem 2. *Let $V \in L^\infty(\mathbb{R}^n)$ take only nonnegative real values, and let $\Omega \subseteq \mathbb{R}^{n-1} \times \mathbb{R}_+$ be a non-empty open set for which the Sobolev embedding $H_0^2(\Omega) \longrightarrow L^2(\Omega)$ is compact. Assume the following:*

I. *$V(\cdot) \asymp e^{-\gamma_0 \langle \cdot \rangle}$ in Ω for some $\gamma_0 \in \mathbb{R}_+$ with $\gamma_0 \ll_n 1$, and V vanishes in $\mathbb{R}^n \setminus \Omega$.*

II. *The complement of Ω in $\mathbb{R}^n$ has a connected interior and is the closure of the interior.*

Then the set of non-scattering energies for V is a discrete subset of $[0, \infty[$ and each of them is of finite multiplicity.

Here the purpose of the condition II is the following: we first prove that the scattered wave u corresponding to a hypothetical non-scattering energy must vanish in the lower half-space. Then the condition II allows us to use the unique continuation principle for the Helmholtz equation to conclude that u vanishes in $\mathbb{R}^n \setminus \Omega$.

The condition on the compact embedding $H_0^2(\Omega) \longrightarrow L^2(\Omega)$ is satisfied for $n \leqslant 3$ if and only if the domain Ω does not contain an infinite sequence of pairwise disjoint congruence balls (see e.g. [1] or Chapter 6 in [2]). For $n \geqslant 4$ the situation is more complicated.

2.3 Rellich type theorems for unbounded domains

Already for the first step, that of reducing the equations which hold in $\mathbb{R}^n$ to equations in the support of the potential, the Rellich type theorem, Theorem 1, must be generalized. In [45] we give two results of this kind: the first is for exponentially decaying inhomogeneities, the second is for polynomially decaying potentials but for domains that are not only contained in a half-space but also grow exponentially thin at infinity. These results are proved with a more traditional complex variables argument [41, 24, 25, 26, 17].

We sketch the proof for the case of exponential decay. The result is as follows.

Theorem 3. *Let* $u \in \mathring{B}_2^*$ *solve*

$$(-\Delta - \lambda)\, u = f,$$

where $\lambda \in \mathbb{R}_+$ *and* $f \in e^{-\gamma_0 \langle \cdot \rangle} L^2(\mathbb{R}^n)$ *for some* $\gamma_0 \in \mathbb{R}_+$, *and suppose that* f *vanishes in* $\mathbb{R}^{n-1} \times \mathbb{R}_-$. *Then also* u *vanishes in* $\mathbb{R}^{n-1} \times \mathbb{R}_-$.

The fundamental idea is to take Fourier transforms, leading to

$$(p - \lambda)\, \widehat{u} = \widehat{f},$$

where $p(\xi) = 4\pi^2 \xi^2$, which holds in $\mathbb{R}^n$. From basic scattering theory, we know that $\widehat{f}$ vanishes on the real sphere

$$M_\lambda^{\mathbb{R}} = \big\{ \xi \in \mathbb{R}^n \;\big|\; p(\xi) = \lambda \big\}.$$

Also, the Fourier transform $\widehat{f}$ extends to an analytic function in

$$D = \big\{ \zeta \in \mathbb{C}^n \;\big|\; |\Im \zeta| < \gamma_0 \big\}.$$

From this, it follows, by flattening the spheres $M_\lambda^{\mathbb{R}}$ and $M_\lambda^{\mathbb{C}}$ locally, that $\widehat{f}$ vanishes on the intersection $D \cap M_\lambda^{\mathbb{C}}$, where $M_\lambda^{\mathbb{C}}$ is the complex sphere

$$M_\lambda^{\mathbb{C}} = \big\{ \zeta \in \mathbb{C}^n \;\big|\; p(\zeta) = \lambda \big\},$$

and furthermore, $\widehat{f}/(p - \lambda)$ extends to an analytic function in D.

Next, fix a point $\xi' \in \mathbb{R}^{n-1}$ with $|\xi'| < \sqrt{\lambda}/2\pi$, and write for simplicity $q(\cdot)$ for $p(\xi', \cdot)$. Now the expression $q(\cdot) - \lambda$ has only two simple zeros $\pm\mu$.

Since f vanishes in $\mathbb{R}^{n-1} \times \mathbb{R}_-$, the Fourier transform $F' f(\xi', \cdot)$ vanishes in $\mathbb{R}_-$, so that the classical half-line Paley–Wiener theorem (see e.g. Thm. 19.2 in

[34]), $\widehat{f}$ has an analytic extension in the last variable to $\mathbb{R} \times i\,]-\infty, \gamma_0[$, and

$$\int_{-\infty}^{\infty} \left|\widehat{f}(\xi', \xi_n - i\eta)\right|^2 \mathrm{d}\xi_n \ll \int_{-\infty}^{\infty} \left|\widehat{f}(\xi', \xi_n)\right|^2 \mathrm{d}\xi_n < \infty$$

for $\eta \in \mathbb{R}_-$. Since $1/(q - \lambda)$ vanishes at infinity uniformly, and since the zeros of $F' f(\xi', \cdot)$ at $\pm\mu$ cancel the simple poles of $1/(q - \lambda)$, also

$$\int_{-\infty}^{\infty} \left|\widehat{u}(\xi', \xi_n - i\eta)\right|^2 \mathrm{d}\xi_n$$

is uniformly bounded by some constant not depending on $\eta \in \mathbb{R}_-$.

Now the one-dimensional half-line Paley–Wiener theorem says that

$$F' u(\xi', x_n) = F_n^{-1} \widehat{u}(\xi', x_n) = 0$$

for all ξ' near the origin and all $x_n \in \mathbb{R}_-$. Since $F'u$ is analytic with respect of the first $n-1$ variables, $F'u(\xi', x_n) = 0$ for all $\xi' \in \mathbb{R}^{n-1}$ and all $x_n \in \mathbb{R}_-$. Finally, taking inverse Fourier transform gives the conclusion that u vanishes in $\mathbb{R}^{n-1} \times \mathbb{R}_-$.

2.4 Some spin-offs of generalizing the Rellich type theorem

Studying the problem of generalizing the Rellich theorem lead to two other interesting things not directly relevant to non-scattering energies. First, we proved a generalized Rellich type theorem where instead of a compactly supported inhomogeneity f, we consider f that is superexponentially decaying and vanishes in a half-space. The conclusion will then again be that the solution u also vanishes in the same half-space. The interesting novelty is that we give a new kind of proof for this result based on real variable techniques, first deriving a Carleman estimate weighted exponentially in one direction from an estimate of Sylvester and Uhlmann [40] and then arguing immediately from it.

As a second aside, and to provide a point of comparison, we present a generalization of the discrete Rellich type theorem of Isozaki and Morioka. A theorem analogous to Theorem 1 also exists for the discrete Laplacian (defined precisely in the paper [45]):

Theorem 4. *Let $u\colon \mathbb{Z}^n \longrightarrow \mathbb{C}$ be a solution to $(-\Delta_{\mathrm{disc}} - \lambda)\, u = f$, where*

$$\frac{1}{R} \sum_{\xi \in \mathbb{Z}^n, |\xi| \leqslant R} |u(\xi)|^2 \longrightarrow 0,$$

as $R \longrightarrow \infty$, and $f \in \ell^2(\mathbb{Z}^n)$ is non-zero only at finitely many points of $\mathbb{Z}^n$, and $\lambda \in]0, n[$. Then u also is non-zero only at finitely many points of $\mathbb{Z}^n$.

This theorem was proved recently by Isozaki and Morioka [19]. A less general version of the result was implicit in the work of Shaban and Vainberg [38].

It turns out that for superexponentially decaying potentials, one gets a much stronger result than in the continuous case: we not only can consider vanishing in half-spaces but vanishing in suitable cones. The proof depends heavily on the arguments in [19] which are first used to show that the solution must be superexponentially decaying. After this, the Rellich type conclusion follows from a repeated application of the definition of the discrete Laplacian.

2.5 Fourth-order operators for non-compactly supported potentials

We can now describe the structure of the proof of Theorem 2. So, assume that $\lambda \in \mathbb{R}_+$ is a non-scattering energy corresponding to total and incident waves $v, w \in B_2^*$, and scattered wave $u = v - w \in \mathring{B}_2^*$. Since

$$(-\Delta - \lambda)\, u = -Vv,$$

in $\mathbb{R}^n$, Theorem 3 guarantees that u vanishes in the lower half-space $\mathbb{R}^{n-1} \times \mathbb{R}_-$ and the condition II of Theorem 2 and the unique continuation principle for the Helmholtz equation guarantees that u vanishes in $\mathbb{R}^n \setminus \Omega$. We therefore end up with the system

$$\begin{cases} (-\Delta + V - \lambda)\, v = 0, \\ (-\Delta - \lambda)\, w = 0, \end{cases}$$

which now holds in Ω. Since V is locally bounded away from zero in Ω, we get for u the fourth-order equation

$$(-\Delta + V - \lambda)\, \frac{1}{V}\, (-\Delta - \lambda)\, u = 0,$$

which again holds in Ω. It turns out that u belongs to a Sobolev space weighted essentially by $V^{-1/2}$. I.e. we have $u \in H_V$, where

$$H_V = \big\{ u \in L_V \;\big|\; \partial^\alpha u \in L_V \text{ for } |\alpha| \leqslant 2 \big\},$$

where in turn

$$L_V = \big\{ u \in L^2_{\mathrm{loc}}(\Omega) \;\big|\; V^{-1/2}\, u \in L^2(\Omega) \big\}.$$

We equip L_V and H_V with the obvious weighted norms.

We shall modify the situation further: the existence of a non-trivial solution $u \in H_V$ to the fourth-order equation is equivalent to the existence of a non-trivial solution $u \in H_V$ to $Q_\lambda(u) = 0$, where Q_λ is the quadratic form

$$Q_\lambda(u) = \int_\Omega \left(-\Delta + V - \overline{\lambda}\right) \overline{u} \cdot \frac{1}{V} \left(-\Delta - \lambda\right) u.$$

This is well-defined for all $\lambda \in \mathbb{C}$ and $u \in H_V$. We let the domain of Q_λ be H_V, and we consider Q_λ as a quadratic form of the Hilbert space L_V.

Our manner of using quadratic forms to establish discreteness is a close relative of the application of quadratic forms to degenerate and singular potentials in the works of Colton, Kirsch and Päivärinta [9], Serov and Sylvester[37], Serov [36], and Hickmann [15].

Now, the discreteness will be obtained just by studying the very basic properties of Q_λ. It is not too hard to verify that Q_λ is in fact something called an entire self-adjoint analytic family of quadratic forms of type (a) with compact resolvent. We recommend Kato's presentation [20] for the related basic theory. The key lemma is the weighted estimate

$$\left\|u\right\|_{H_V} \asymp_\lambda \left\|\left(-\Delta - \lambda\right) u\right\|_{L_V} + \left\|u\right\|_{L_V},$$

true for all $u \in H_V$ and any $\lambda \in \mathbb{R}_+$, which is established using arguments from Appendix A of [3].

By the theory of quadratic forms and analytic perturbation theory, each Q_λ corresponds to a unique closed operator T_λ of L_V and we immediately get certain excellent properties for T_λ. In particular, for $\lambda \in \mathbb{R}$, the quadratic form $Q_\lambda(u)$ corresponds to a unique self-adjoint operator T_λ of the Hilbert space L_V with compact resolvent, and the above fourth-order equation has a non-trivial H_V-solution if and only if 0 is an eigenvalue of T_λ. Furthermore, the eigenvalues of T_λ are given, including multiplicity, by a sequence $\mu_1(\lambda)$, $\mu_2(\lambda)$, $\ldots$ of functions on $\mathbb{R}$, which depend real-analytically on λ, and when λ is changed by some finite amount δ, the eigenvalues $\mu_\ell(\lambda)$ can each change by at most a constant which is independent of ℓ and only depends on δ (and, naturally, V).

So, we have established that non-scattering energies lead to zeros of $\mu_\ell(\lambda)$. Now the discreteness follows immediately from the observation that $Q_\lambda(u) > 0$ for all $\lambda \in \mathbb{R}_-$ and $u \in H_V \setminus 0$, as this means that none of the functions $\mu_\ell(\lambda)$ can vanish identically.

3 Corner scattering

Our second major topic is generalizing the non-existence of non-scattering energies to potentials with corners. The first such result only considers rectangular corners [5].

The novel approach introduced in [5] begins by assuming that a non-scattering energy exists with the intention to derive a contradiction. To illustrate the ideas, we assume that

$$\begin{cases} (-\Delta + V - \lambda)\, v = 0, \\ (-\Delta - \lambda)\, w = 0 \end{cases}$$

in $\mathbb{R}^2$, where $v, w \in B_2^*(\mathbb{R}^2)$ and $u = v - w \in \mathring{B}_2^*(\mathbb{R}^2)$. For simplicity, we assume here that the potential V is assumed to be supported in a closed sector $C \subseteq \mathbb{R}^2$ of angle smaller than π with vertex at the origin, and to be, say, smooth in C, compactly supported, and nonzero at the origin. Our paper [29] discusses more general situations.

The plan is to study the function w near the origin. As w is real-analytic, we may expand it as Taylor series in a neighbourhood of the origin, and pick the lowest degree nonzero terms, which form a harmonic homogeneous polynomial $H(x) \not\equiv 0$ of degree $N \geqslant 0$. The sought-for contradiction will come in the form $H(x) \equiv 0$.

It turns out that

$$\int\limits_C V(x)\, w(x)\, \widetilde{w}(x)\, \mathrm{d}x = 0$$

for any solution $\widetilde{w} \in H^2_{\mathrm{loc}}(x)$ to

$$(-\Delta + V - \lambda)\, \widetilde{w} = 0$$

in $\mathbb{R}^n$. A major component of [5] is constructing complex geometrical optics solutions of the form $\widetilde{w} = e^{-\rho\cdot x}(1 + \psi(x))$ for $\rho \in \mathbb{C}^n$ with $\rho \cdot \rho = 0$ and with ψ satisfying pleasant L^p estimates. In [5], such solutions were constructed in all dimensions $n \geq 2$ but for "polygonal" cones C (that is, the cross-section of the cone is a polygon). Our three dimensional result is used for circular cones, and so we base our CGO construction on certain L^p estimates from [21]. This argument gives sufficient estimates for $n \in \{2, 3\}$.

Thus, the CGO solutions are obtained here differently, and furthermore for ρ with $\rho \cdot \rho = \lambda$. Substituting these solutions to the the equality involving $\int_C$ gives, after some detailed estimations,

$$\int\limits_C e^{-\rho\cdot x}\, H(x)\, \mathrm{d}x \ll |\rho|^{-N-2-\beta}\,,$$

for some small $\beta \in \mathbb{R}_+$, as $|\rho| \longrightarrow \infty$, and we restrict to ρ such that, say, $\Re\rho \cdot x \geqslant \varepsilon > 0$ for all $x \in C$ for some fixed $\varepsilon \in \mathbb{R}_+$. On the other hand, by the homogeneity of $H(x)$, we have

$$\int\limits_C e^{-\rho\cdot x}\, H(x)\,\mathrm{d}x = |\rho|^{-N-2} \int\limits_C e^{-\rho/|\rho|\cdot x}\, H(x)\,\mathrm{d}x,$$

and this is compatible with the previous estimate only if

$$\int\limits_C e^{-\rho\cdot x}\, H(x)\,\mathrm{d}x = 0$$

for certain $\rho \in \mathbb{C}^n$ with $\rho \cdot \rho = 0$ (as opposed to $\rho \cdot \rho = \lambda$).

At this point, our line of reasoning irreversibly departs from that of [5]. The main novelty in our paper [29] is based on moving here to polar coordinates, which turns out to reduce the dimension and lead to integrals over $C \cap S^{n-1}$. In two dimensions, we conclude that

$$\int\limits_{C\cap S^1} (\rho\cdot\vartheta)^{-N-2}\, H(\vartheta)\,\mathrm{d}\vartheta = 0$$

for the same ρ as before, and where $\mathrm{d}\vartheta$ is the obvious measure on the unit circle $S^1 \subseteq \mathbb{R}^2$.

Since $H(x)$ was harmonic, it must be of the form

$$a\,(x_1 + ix_2)^N + b\,(x_1 - ix_2)^N$$

for some constants a and b. Now, choosing ρ suitably, the vanishing of the last integrals will lead, through some explicit calculations, to $a = b = 0$, establishing the desired contradiction.

In three dimensions, the same approach can largely be executed for circular cones. Now $H(x)$ will be a linear combination of spherical harmonics, and the vanishing of the coefficients will follow if certain integrals depending on the angle of the cone do not vanish. However, these integrals do not anymore seem to allow explicit evaluation. Yet, they depend analytically on the angle, and we are able to prove that none of them is identically zero, and this is enough for the conclusion that circular conical corners prohibit non-scattering energies except possibly for some at most countable set of exceptional angles.

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